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The number of eigenvalues of three-particle Schrödinger operators on lattices

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Abstract

We consider the Hamiltonian of a system of three quantum mechanical particles (two identical fermions and a boson) on the three-dimensional lattice $\mathbb{Z}^3$ interacting by means of zero-range attractive potentials. We describe the location and structure of the essential spectrum of the three-particle discrete Schrödinger operator $H_\gamma(K)$, K being the total quasi-momentum belonging to the three-dimensional torus $\mathbb{T}^3 = (-\pi, \pi]^3$ and $\gamma > 0$ the ratio of the mass of fermion to boson. We choose for $\gamma > 0$ the interaction $\mu(\gamma)$ in such a way that the system consisting of one fermion and one boson has a zero-energy resonance. For all nonzero values of the quasi-momentum $K \in \mathbb{T}^3$, we prove the finiteness of the number $N(K, \gamma; \tau_\gamma(K))$ of eigenvalues of $H_\gamma(K)$ below the bottom $\tau_\gamma(K)$ of the essential spectrum and we give for $N(K, \gamma; 0)$ an asymptotics as $K \to 0$. Moreover, we prove the existence of infinitely many eigenvalues of the operator $H_\gamma(0)$ and give for the number $N(0, \gamma; z)$ of eigenvalues lying below $z < 0$ an asymptotics as $z \to 0$.

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1. Introduction

We consider a system of three particles (two identical fermions and a boson) on the three-dimensional lattice $\mathbb{Z}^3$ interacting by means of zero-range pairs attractive potentials.

The main goal of the present paper is to prove (under the relevant conditions) the finiteness or infiniteness of the number of eigenvalues lying below the bottom of the essential spectrum of the three-particle discrete Schrödinger operator $H_\gamma(K)$ depending on the total quasi-momentum $K \in \mathbb{T}^3 = (-\pi, \pi]^3$ and the ratio $\gamma > 0$ of the mass of the fermion to the boson.

Efimov's effect is one of the remarkable results in the spectral analysis for continuous and discrete three-particle Schrödinger operators: if none of the three two-particle Schrödinger operators (corresponding to the two-particle subsystems) has negative eigenvalues, but at least two of them have a zero-energy resonance, then this three-particle Schrödinger operator has an infinite number of discrete eigenvalues, accumulating at zero.

Since its discovery by Efimov in 1970 [1], much research have been devoted to this subject. See, for example, [2–15]. The main result obtained by Sobolev [8] (see also [10]) is an asymptotics of the form $\mathcal{U}_0|\log|\lambda||$ for the number of eigenvalues below $\lambda < 0$, where the coefficient $\mathcal{U}_0$ does not depend on the two-particle potentials v_α and is a positive function of the ratios $m_1/m_2, m_2/m_3$ of the masses of the three particles.

The existence of Efimov's effect for N-body quantum systems with $N \geqslant 4$ has been proved by Wang in [16].

In fact in [16] for the total (reduced) Hamiltonian a lower bound on the number of eigenvalues of the form $C_0|\log(E_0 - \lambda)|$ is given, when λ tends to E_0, where C_0 is a positive constant and E_0 is the bottom of the essential spectrum.

The kinematics of quantum particles on lattices, even in the two- and three-particle sectors, is rather exotic. For instance, due to the fact that the discrete analogues of the Laplacian or its generalizations (see (2.1)) are not translationally invariant, the Hamiltonian of a system does not separate into two parts, one relating to the center-of-mass motion and the other one relating to the internal degrees of freedom.

As a consequence, any local substitute of the effective mass-tensor (of a ground state) depends on the quasi-momentum of the system and, in addition, it is only semi-additive (with respect to the partial order on the set of positive definite matrices). This is the so-called *excess mass* phenomenon for lattice systems (see, e.g., [17, 18]): the effective mass of the bound state of an N-particle system is greater than and, in general, not equal to the sum of the effective masses of the constituent quasi-particles.

The three-body problem on lattices can be reduced to effective three-particle Schrödinger operators by using the Gelfand transform. The underlying Hilbert space $\ell_2((\mathbb{Z}^d)^3)$ is decomposed as a direct von Neumann integral associated with the representation of the discrete group $\mathbb{Z}^3$ by shift operators on the lattice, and the total three-body Hamiltonian appears to be decomposable. In contrast to the continuous case, the corresponding fiber Hamiltonians $H_\gamma(K)$ associated with the direct decomposition depend parametrically on the quasi-momentum $K \in \mathbb{T}^3$, which ranges over a cell of the dual lattice. Due to the loss of the spherical symmetry of the problem, the spectra of the family $H_\gamma(K)$ turn out to be rather sensitive to the quasi-momentum K.

In particular, Efimov's effect exists only for the zero value of the three-particle quasi-momentum K (see, e.g., [15, 19] for relevant discussions and [17, 18, 21–25] for the general study of the low-lying excitation spectrum for quantum systems on lattices).

Denote by $\tau_\gamma(K)$ the bottom of the essential spectrum of the three-particle discrete Schrödinger operator $H_\gamma(K)$, $K \in \mathbb{T}^3$ and by $N(K, \gamma; z)$ the number of eigenvalues lying below $z \leqslant \tau_\gamma(K)$.

The main results of the present paper are as follows:

(i) for any $\gamma > 0$ and $K \in \mathbb{T}^3 \setminus \{0\}$, the number $N(K, \gamma; \tau_\gamma(K))$ is finite and the following asymptotics holds:

$$\lim_{|K| \to 0} \frac{N(K, \gamma; 0)}{|\log|K||} = 2U(\gamma), \qquad (0 < U(\gamma) < \infty);$$

(ii) for all $\gamma > 0$ the operator $H_\gamma(0)$ has infinitely many eigenvalues below the bottom of the essential spectrum and for the number $N(0,\gamma;z)$, $z < 0$, the asymptotics

$$\lim_{z\to -0} \frac{N(0,\gamma;z)}{|\log|z||} = U(\gamma)$$

holds, which is similar to the asymptotics found in the continuous case by Sobolev in [8].

The result (i) is characteristic for lattice systems and does not have any analogue in the continuous case.

We underline that these results are in contrast to similar results for the continuous three-particle Schrödinger operators, where the number of eigenvalues does not depend on the three-particle total momentum $K \in R^3$ [26].

These results are also in contrast to the results for two-particle Schrödinger operators on lattices (see [27]), which have finitely many eigenvalues for all $k \in U_\delta(0) = \{k \in \mathbb{T}^3 : |k| < \delta\}$.

We note that the operator $H_\gamma(K)$ has been considered before, but only the existence of infinitely many eigenvalues below the bottom of the essential spectrum of $H_\gamma(0)$ has been announced in [28] .

The organization of the present paper is as follows.

Section 1 is an introduction. In section 2 we introduce the Hamiltonians of systems of two and three particles in coordinate and momentum representations as bounded self-adjoint operators in the corresponding Hilbert spaces. In section 3 we introduce the total quasi-momentum and decompose the energy operators into von Neumann direct integrals, choosing relative coordinate systems. In section 4, we state the main results of the paper. In section 5, we prove the existence of a unique positive eigenvalue below the bottom of the essential spectrum of the two-particle discrete Schrödinger operator $\hbar_\gamma(k)$, $k \in \mathbb{T}^3$ (theorem 4.7). In section 6, we introduce the 'channel operators' and describe the essential spectrum of $H_\gamma(K)$ by the spectrum of $\hbar_\gamma(k)$, $k \in \mathbb{T}^3$ (theorem 4.8). In section 7 we give the Birman–Schwinger principle for $H_\gamma(K)$ (theorem 7.2), and in section 8 we prove the finiteness of $N(K,\gamma;\tau_\gamma(K))$, $K \in \mathbb{T}^3\backslash\{0\}$ (theorem 4.9). In section 9, we derive the asymptotics for the numbers $N(0,\gamma;z)$ and $N(K,\gamma;0)$ as $z \to 0$ and $|K| \to 0$ respectively (theorem 4.9).

Throughout the present paper, we adopt the following notations. We denote by $\mathbb{T}^3$ the three-dimensional torus, i.e. the cube $(-\pi,\pi]^3$ with appropriately identified sides. The torus $\mathbb{T}^3$ will always be considered as an Abelian group with respect to the addition and multiplication by real numbers regarded as operations on the three-dimensional space $\mathbb{R}^3$ modulo $(2\pi\mathbb{Z})^3$. We also denote by $\mathbb{T}_0^3$ the torus $\mathbb{T}^3$ without the point $0 \in \mathbb{T}^3$, i.e. $\mathbb{T}_0^3 \equiv \mathbb{T}^3\backslash\{0\}$.

For each $\delta > 0$, the notation $U_\delta(0) = \{K \in \mathbb{T}^3 : |K| < \delta\}$ stands for a δ-neighborhood of the origin and $U_\delta^0(0) = U_\delta(0)\backslash\{0\}$ for a punctured δ-neighborhood. The subscripts α and β always equal to 1 or 2 and $\alpha \neq \beta$.

2. Description of the energy operators of two and three particles on the lattice $\mathbb{Z}^3$

Let $\mathbb{Z}^3$ be the three-dimensional lattice and let $(\mathbb{Z}^3)^m$, $m \in N$ be the Cartesian mth power of $\mathbb{Z}^3$. Denote by $\ell_2((\mathbb{Z}^3)^3)$ the Hilbert space of square-summable functions $\hat{\varphi}$ defined on $(\mathbb{Z}^3)^3$ and let $\ell_2^a((\mathbb{Z}^3)^3) \subset \ell_2((\mathbb{Z}^3)^3)$ be the subspace of functions antisymmetric with respect to the first two coordinates.

We consider a system of three particles (two identical fermions and one boson) moving on the three-dimensional lattice $\mathbb{Z}^3$. Each of the fermion interacts with the boson via a zero-range pair attractive potential with interaction energy $\mu > 0$. The free Hamiltonian $\widehat{H}_\gamma^0$ of this system in the coordinate representation is an operator on the Hilbert space $\ell_2^a((\mathbb{Z}^3)^3)$ of the form

$$\big(\widehat{H}_\gamma^0\hat{\psi}\big)(x_1,x_2,x_3)=\sum_{s\in\mathbb{Z}^3}\hat{\varepsilon}(s)[\hat{\psi}(x_1+s,x_2,x_3)+\hat{\psi}(x_1,x_2+s,x_3)+\gamma\hat{\psi}(x_1,x_2,x_3+s)],$$
$$\hat{\psi}\in\ell_2^a((\mathbb{Z}^3)^3), \tag{2.1}$$

where the function $\hat{\varepsilon}(s)$ is defined on $\mathbb{Z}^3$ by

$$\hat{\varepsilon}(s)=\begin{cases}3, & s=0\\ -\frac{1}{2}, & |s|=1\\ 0, & \text{otherwise,}\end{cases} \tag{2.2}$$

the number $\gamma>0$ being the ratio of the mass of the fermion to that of the boson.

It is clear that the free Hamiltonian (2.1) is a bounded self-adjoint operator on $\ell_2^a((\mathbb{Z}^3)^3)$.

The three-particle Hamiltonian $\widehat{H}_{\mu,\gamma}$ of a quantum-mechanical three-particle system is a bounded perturbation of the free Hamiltonian $\widehat{H}_\gamma^0$:

$$\widehat{H}_{\mu,\gamma}=\widehat{H}_\gamma^0-\mu(\widehat{V}_1+\widehat{V}_2+\widehat{V}_3). \tag{2.3}$$

Here $\widehat{V}_\alpha=\widehat{V},\alpha=1,2$, is the multiplication operator on $\ell_2^a((\mathbb{Z}^3)^3)$ given by

$$\begin{aligned}(\widehat{V}_\alpha\hat{\psi})(x_1,x_2,x_3)&=\delta_{x_\alpha x_3}\hat{\psi}(x_1,x_2,x_3), &&\hat{\psi}\in\ell_2^a((\mathbb{Z}^3)^3),\\ (\widehat{V}_3\hat{\psi})(x_1,x_2,x_3)&=\delta_{x_1x_2}\hat{\psi}(x_1,x_2,x_3), &&\hat{\psi}\in\ell_2^a((\mathbb{Z}^3)^3),\end{aligned} \tag{2.4}$$

where δ_{xy} is the Kroneker delta.

The Hamiltonian $\hat{h}_{\mu,\gamma}$ describing the two-particle subsystem consisting of a fermion and a boson, which interacts via zero-range pair attractive potentials, is introduced as the bounded self-adjoint operator on the Hilbert space $\ell_2((\mathbb{Z}^3)^2)$ given by

$$\hat{h}_{\mu,\gamma}=\hat{h}_\gamma^0-\mu\hat{v}, \tag{2.5}$$

where

$$\big(\hat{h}_\gamma^0\hat{\psi}\big)(x_\alpha,x_3)=\sum_{s\in\mathbb{Z}^3}\hat{\varepsilon}(s)[\hat{\psi}(x_\alpha+s,x_3)+\gamma\hat{\psi}(x_\alpha,x_3+s)],\qquad \hat{\psi}\in\ell_2((\mathbb{Z}^3)^2),\quad \alpha=1,2,$$

and

$$(\hat{v}\hat{\psi})(x_\alpha,x_3)=\delta_{x_\alpha x_3}\hat{\psi}(x_\alpha,x_3),\qquad \hat{\psi}\in\ell_2((\mathbb{Z}^3)^2).$$

Let $\ell_2^a((\mathbb{Z}^3)^2)\subset\ell_2((\mathbb{Z}^3)^2)$ be the subspace of antisymmetric functions. The Hamiltonian $\hat{h}_\mu$ describing the two-particle subsystem consisting of two identical fermions interacting via zero-range pair potentials, with interaction energy $\mu>0$, is a bounded self-adjoint operator on $\ell_2^a((\mathbb{Z}^3)^2)$ given by

$$\hat{h}_\mu=\hat{h}_1^0-\mu\hat{v}_3, \tag{2.6}$$

where

$$\big(\hat{h}_1^0\hat{\psi}\big)(x_1,x_2)=\sum_{s\in\mathbb{Z}^3}\hat{\varepsilon}(s)[\hat{\psi}(x_1+s,x_2)+\hat{\psi}(x_1,x_2+s)],\qquad \hat{\psi}\in\ell_2^a((\mathbb{Z}^3)^2)$$

and

$$(\hat{v}_3\hat{\psi})(x_1,x_2)=\delta_{x_1x_2}\hat{\psi}(x_1,x_2),\qquad \hat{\psi}\in\ell_2^a((\mathbb{Z}^3)^2).$$

Remark 2.1. We remark that for any $\hat{\psi}\in\ell_2^a((\mathbb{Z}^3)^2)$, the relations

$$(\hat{v}_3\hat{\psi})(x_1,x_2)=\delta_{x_1x_2}\hat{\psi}(x_1,x_2)=-\delta_{x_1x_2}\hat{\psi}(x_1,x_2)=-(\hat{v}_3\hat{\psi})(x_1,x_2)$$

hold, and hence the operator v_3 on $\ell_2^a((\mathbb{Z}^3)^2)$ vanishes.

Similarly, for the operator $\widehat{V}_3$, defined in (2.4), for any $\hat{\psi} \in \ell_2^a((\mathbb{Z}^3)^3)$ the following equalities

$$(\widehat{V}_3\hat{\psi})(x_1, x_2, x_3) = \delta_{x_1x_2}\hat{\psi}(x_1, x_2, x_3) = -\delta_{x_1x_2}\hat{\psi}(x_1, x_2, x_3) = -(\widehat{V}_3\hat{\psi})(x_1, x_2, x_3)$$

hold, and hence $\widehat{V}_3$ on $\ell_2^a((\mathbb{Z}^3)^3)$ vanishes.

2.1. The momentum representation

Denote by $L^2((\mathbb{T}^3)^m)$ the Hilbert space of square-integrable functions φ defined on $(\mathbb{T}^3)^m$, where $(\mathbb{T}^3)^m$ denotes the Cartesian mth power of $\mathbb{T}^3 = (-\pi, \pi]^3$.

Let $L^{2,a}((\mathbb{T}^3)^3)$ be the subspace of $L^2((\mathbb{T}^3)^3)$ consisting of functions which are antisymmetric in the first two coordinates.

Let $\mathcal{F}_m : L^2((\mathbb{T}^3)^m) \to \ell_2((\mathbb{Z}^3)^m)$, $m \in \mathbb{N}$, denote the standard Fourier transform.

Denote by $\mathcal{F}_3^a$ the restriction of $\mathcal{F}_3$ to the subspace $L^{2,a}((\mathbb{T}^3)^3)$. It is easy to check that

$$\mathcal{F}_3^a : L^{2,a}((\mathbb{T}^3)^3) \to \ell_2^a((\mathbb{Z}^3)^3).$$

In the momentum representation, the three- and two-particle Hamiltonians are respectively given by the bounded self-adjoint operators

$$H_{\mu,\gamma} = (\mathcal{F}_3^a)^{-1}\widehat{H}_{\mu,\gamma}\mathcal{F}_3^a, \qquad h_{\mu,\gamma} = (\mathcal{F}_2)^{-1}\hat{h}_{\mu,\gamma}\mathcal{F}_2$$

on the Hilbert space $L^{2,a}((\mathbb{T}^3)^3)$, respectively $L^2((\mathbb{T}^3)^2)$.

The three-particle Hamiltonian $H_{\mu,\gamma}$ (in the momentum representation) is of the form

$$H_{\mu,\gamma} = H_\gamma^0 - \mu(V_1 + V_2), \tag{2.7}$$

where

$$\big(H_\gamma^0 f\big)(k_1, k_2, k_3) = (\varepsilon(k_1) + \varepsilon(k_2) + \gamma\varepsilon(k_3))f(k_1, k_2, k_3),\ f \in L^{2,a}((\mathbb{T}^3)^3), \tag{2.8}$$

$$\begin{aligned}(V_\alpha f)(k_1, k_2, k_3) &= (Vf)(k_1, k_2, k_3)\\ &= \frac{1}{(2\pi)^3}\int_{(\mathbb{T}^3)^3}\delta(k_\alpha - k'_\alpha)\delta(k_\beta + k_3 - k'_\beta - k'_3)f(k'_1, k'_2, k'_3)\,\mathrm{d}k'_1\,\mathrm{d}k'_2\,\mathrm{d}k'_3,\end{aligned} \tag{2.9}$$

$f \in L^{2,a}((\mathbb{T}^3)^3)$, $\alpha, \beta = 1, 2$, $\alpha \neq \beta$. The function ε is of the form

$$\varepsilon(p) = \sum_{i=1}^{3}(1 - \cos p^{(i)}), \qquad p = (p^{(1)}, p^{(2)}, p^{(3)}) \in \mathbb{T}^3, \tag{2.10}$$

and $\delta(k)$ denotes the three-dimensional Dirac delta-function.

The two-particle Hamiltonian $h_{\mu,\gamma}$ (in the momentum representation) is of the form

$$h_{\mu,\gamma} = h_\gamma^0 - \mu v, \tag{2.11}$$

where

$$\big(h_\gamma^0 f\big)(k_\alpha, k_3) = (\varepsilon(k_\alpha) + \gamma\varepsilon(k_3))f(k_\alpha, k_3), \qquad \alpha = 1, 2, \quad f \in L^2((\mathbb{T}^3)^2), \tag{2.12}$$

and the operator v can be written in the form

$$\begin{aligned}(vf)(k_\alpha, k_3) &= (v_\alpha f)(k_\alpha, k_3)\\ &= \frac{1}{(2\pi)^3}\int_{(\mathbb{T}^3)^2}\delta(k_\alpha + k_3 - k'_\alpha - k'_3)f(k'_\alpha, k'_3)\,\mathrm{d}k'_\alpha\,\mathrm{d}k'_3,\\ &\alpha = 1, 2, \qquad f \in L^2((\mathbb{T}^3)^2).\end{aligned}$$

Remark 2.2. We remark that the two-particle operator h_μ associated with the two-particle Hamiltonian of a system of two identical fermions (in the momentum representation) acts in the Hilbert space $L^{2,a}((\mathbb{T}^3)^2) \subset L^2((\mathbb{T}^3)^2)$ of all antisymmetric functions on $(\mathbb{T}^3)^2$ and according to remark 2.1 is of the form

$$(h_\mu f)(k_1,k_2) = \big(h_1^0 f\big)(k_1,k_2) = (\varepsilon(k_1)+\varepsilon(k_2)) f(k_1,k_2), \qquad f \in L^{2,a}((\mathbb{T}^3)^2).$$

3. Decomposition of the energy operators into von Neumann direct integrals. Quasi-momentum and coordinate systems

Given $m \in \mathbb{N}$, denote by $\widehat{U}_s^m$, $s \in \mathbb{Z}^3$, the unitary operators on the Hilbert space $\ell_2((\mathbb{Z}^3)^m)$ defined by

$$\big(\widehat{U}_s^m f\big)(n_1,n_2,\ldots,n_m) = f(n_1+s,n_2+s,\ldots,n_m+s), \qquad f \in \ell_2((\mathbb{Z}^3)^m).$$

We easily see that

$$\widehat{U}_{s+p}^m = \widehat{U}_s^m \widehat{U}_p^m, \qquad s,p \in \mathbb{Z}^3,$$

that is, $\widehat{U}_s^m$, $s \in \mathbb{Z}^3$, is a unitary representation of the Abelian group $\mathbb{Z}^3$ in $\ell_2((\mathbb{Z}^3)^3)$.

Denote by $\widehat{U}_{as}^3$ the restriction of $\widehat{U}_s^3$, $s \in \mathbb{Z}^3$, to the subspace $\ell_2^a((\mathbb{T}^3)^3)$.

Via the Fourier transform $\mathcal{F}_3^a$, the unitary representation $\widehat{U}_{as}^3$ of the group $\mathbb{Z}^3$ in $\ell_2^a((\mathbb{Z}^3)^3)$ induces the representation U_{as}^3 of the group $\mathbb{Z}^3$ in the Hilbert space $L^{2,a}((\mathbb{T}^3)^3)$ by the unitary (multiplication) operators

$$\big(U_{as}^3 f\big)(k_1,k_2,k_3) = \exp(-\mathrm{i}(s,k_1+k_2+k_3)) f(k_1,k_2,k_3), \qquad s \in \mathbb{Z}^3, \quad f \in L^{2,a}((\mathbb{T}^3)^3).$$

For any $K \in \mathbb{T}^3$, we define $\mathbb{F}_K^3$ as follows:

$$\mathbb{F}_K^3 = \{(k_1,k_2,K-k_1-k_2) \in (\mathbb{T}^3)^3 : k_1,k_2 \in \mathbb{T}^3\}.$$

Let $L^{2,a}\big(\mathbb{F}_K^3\big)$ be the subspace of $L^2\big(\mathbb{F}_K^3\big)$ consisting of functions which are antisymmetric in the first two coordinates.

The Hamiltonian $H_{\mu,\gamma}$ commutes with the group U_{as}^3, $s \in \mathbb{Z}^3$, and hence the decomposition

$$L^{2,a}((\mathbb{T}^3)^3) = \int_{K\in\mathbb{T}^3} \oplus L^{2,a}\big(\mathbb{F}_K^3\big)\,\mathrm{d}K$$

yields the decompositions

$$U_{as}^3 = \int_{K\in\mathbb{T}^3} \oplus U_{as}^3(K)\,\mathrm{d}K, \qquad H_{\mu,\gamma} = \int_{K\in\mathbb{T}^3} \oplus \widetilde{H}_{\mu,\gamma}(K)\,\mathrm{d}K,$$

of U_{as}^3, $s \in \mathbb{Z}^3$, and $H_{\mu,\gamma}$ respectively, where

$$U_{as}^3(K) = \exp(-\mathrm{i}(s,K)) \times I \qquad \text{on} \quad L^{2,a}\big(\mathbb{F}_K^3\big),$$

and $I = I_{L^{2,a}(\mathbb{F}_K^3)}$ denotes the identity operator on the Hilbert space $L^{2,a}\big(\mathbb{F}_K^3\big)$.

The Hamiltonian $h_{\mu,\gamma}$ commutes with the group U_s^2, $s \in \mathbb{Z}^3$, and hence $h_{\mu,\gamma}$ can be decomposed into the direct integral

$$h_{\mu,\gamma} = \int_{k\in\mathbb{T}^3} \oplus \tilde{h}_{\mu,\gamma}(k)\,\mathrm{d}k,$$

with respect to the decomposition

$$L^2((\mathbb{T}^3)^2) = \int_{k\in\mathbb{T}^3} \oplus L^2\big(\mathbb{F}_K^2\big)\,\mathrm{d}k,$$

where

$$\mathbb{F}_K^2 = \{(k_1, k-k_1)\in(\mathbb{T}^3)^2 : k_1 \in \mathbb{T}^3\}.$$

Introduce the mappings

$$\pi^{(3)}_{\alpha 3} : (\mathbb{T}^3)^3 \to (\mathbb{T}^3)^2, \qquad \pi^{(3)}_{\alpha 3}((k_\alpha, k_\beta, k_3)) = (k_\alpha, k_3), \qquad \alpha \neq \beta, \quad \alpha, \beta = 1, 2,$$

and

$$\pi^{(2)}_{\alpha} : (\mathbb{T}^3)^2 \to \mathbb{T}^3, \qquad \pi^{(2)}_{\alpha}((k_\alpha, k_3)) = k_\alpha, \quad \alpha = 1, 2.$$

Denote by $\pi^{(3)}_K$, $K \in \mathbb{T}^3$, and $\pi^{(2)}_k$, $k \in \mathbb{T}^3$, the restrictions of $\pi^{(3)}_{\alpha 3}$ and $\pi^{(2)}_{\alpha}$ onto $\mathbb{F}_K^3 \subset (\mathbb{T}^3)^3 \mathbb{F}_K^2 \subset (\mathbb{T}^3)^2$, respectively, that is,

$$\pi^{(3)}_K = \pi^{(3)}_{\alpha 3}\big|_{\mathbb{F}_K^3} \qquad \text{and} \qquad \pi^{(2)}_k = \pi^{(2)}_{\alpha}\big|_{\mathbb{F}_K^2}. \tag{3.1}$$

At this point it is useful to remark that $\mathbb{F}_K^3$, $K \in \mathbb{T}^3$, and $\mathbb{F}_k^2$, $k \in \mathbb{T}^3$, are six- and three-dimensional manifolds isomorphic to $(\mathbb{T}^3)^2$ and $\mathbb{T}^3$, respectively.

The proof of the following lemma is evident.

Lemma 3.1. *The mappings $\pi^{(3)}_K$, $K \in \mathbb{T}^3$, and $\pi^{(2)}_k$, $k \in \mathbb{T}^3$, are bijective from $\mathbb{F}_K^3 \subset (\mathbb{T}^3)^3$ and $\mathbb{F}_k^2 \subset (\mathbb{T}^3)^2$ onto $(\mathbb{T}^3)^2$ and $\mathbb{T}^3$, with the inverse mappings given by*

$$\left(\pi^{(3)}_K\right)^{-1}(k_\alpha, k_3) = (k_\alpha, k_3, K - k_\alpha - k_3), \qquad \alpha = 1, 2,$$
$$\left(\pi^{(2)}_k\right)^{-1}(k_\alpha) = (k_\alpha, k - k_\alpha), \qquad \alpha = 1, 2$$

respectively.

Let

$$U_K : L^2\left(\mathbb{F}_K^3\right) \to L^2((\mathbb{T}^3)^2), \qquad U_K f = f \circ \left(\pi^{(3)}_K\right)^{-1}, \qquad K \in \mathbb{T}^3,$$

and

$$u_k : L^2\left(\mathbb{F}_k^2\right) \to L^2(\mathbb{T}^3), \qquad u_k g = g \circ \left(\pi^{(2)}_k\right)^{-1}, \qquad k \in \mathbb{T}^3,$$

where $\pi^{(3)}_K$ and $\pi^{(2)}_k$ are defined by (3.1). Then U_K and u_k are unitary operators, and the following equalities hold:

$$H_{\mu,\gamma}(K) = U_K \widetilde{H}_{\mu,\gamma}(K)(U_K)^{-1}, \qquad h_{\mu,\gamma}(k) = u_k \tilde{h}_{\mu,\gamma}(k)(u_k)^{-1}. \tag{3.2}$$

The two-particle Schrödinger operators $h_{\mu,\gamma}(k)$, $k \in \mathbb{T}^3$, are of the form

$$h_{\mu,\gamma}(k) = h^0_\gamma(k) - \mu v, \tag{3.3}$$

where

$$h^0_\gamma(k) f(q) = \mathcal{E}_{k,\gamma}(q) f(q), \qquad f \in L^2(\mathbb{T}^3), \tag{3.4}$$

$$\mathcal{E}_{k,\gamma}(q) = \varepsilon(q) + \gamma\varepsilon(k - q), \tag{3.5}$$

$$(vf)(q) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} f(q')\, \mathrm{d}q', \qquad f \in L^2(\mathbb{T}^3). \tag{3.6}$$

Remark 3.2. We remark that the two-particle Schrödinger operators $h_\mu(k)$, $k \in \mathbb{T}^3$, associated with the two-particle Hamiltonian h_μ of a system of two identical fermions act in the Hilbert space $L^{2,o}(\mathbb{T}^3) \subset L^2(\mathbb{T}^3)$ of all odd functions on $\mathbb{T}^3$ and according to remark 2.2 are of the form

$$h_\mu(k) f(q) = (\varepsilon(q) + \varepsilon(k - q)) f(q), \qquad f \in L^{2,o}(\mathbb{T}^3).$$

The three-particle Schrödinger operators $H_{\mu,\gamma}(K)$, $K \in \mathbb{T}^3$, from (3.2) are of the form

$$H_{\mu,\gamma}(K) = H_\gamma^0(K) - 2\mu V. \tag{3.7}$$

The operators $H_\gamma^0(K)$ and V in the coordinates (k_α, k_3) are defined by

$$\begin{aligned} \big(H_\gamma^0(K) f\big)(k_\alpha, k_3) &= E(K, \gamma; k_\alpha, k_3) f(k_\alpha, k_3), && f \in L^2((\mathbb{T}^3)^2), \\ (Vf)(k_\alpha, k_3) = (V_\alpha f)(k_\alpha, k_3) &= \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} f(k_\alpha, k_3')\, \mathrm{d}k_3', && f \in L^2((\mathbb{T}^3)^2), \end{aligned} \tag{3.8}$$

where

$$E(K, \gamma; k_\alpha, k_3) = \varepsilon(k_\alpha) + \varepsilon(k_3) + \gamma \varepsilon(K - k_\alpha - k_3), \qquad k_\alpha, k_3 \in \mathbb{T}^3. \tag{3.9}$$

Remark 3.3. The decomposition

$$L^2((\mathbb{T}^3)^2) = L^2(\mathbb{T}^3) \otimes L^2(\mathbb{T}^3)$$

of the Hilbert space $L^2((\mathbb{T}^3)^2)$ yields the representation $V = I \otimes v$ for the operator V, where v is the two-particle interaction operator defined by (3.6).

4. Statement of the main results

In this section, we give precise formulations of the main results. The perturbation v of the multiplication operator $h_\gamma^0(k)$ is a bounded self-adjoint operator of rank one. Therefore in accordance to Weyl's theorem, the essential spectrum of $h_{\mu,\gamma}(k)$, $k \in \mathbb{T}^3$, fills the following interval on the real axis:

$$\sigma_{\mathrm{ess}}(h_{\mu,\gamma}(k)) = [\mathcal{E}_{\min,\gamma}(k), \mathcal{E}_{\max,\gamma}(k)],$$

where

$$\mathcal{E}_{\min,\gamma}(k) = \min_{p\in\mathbb{T}^3} \mathcal{E}_{k,\gamma}(p), \qquad \mathcal{E}_{\max,\gamma}(k) = \max_{p\in\mathbb{T}^3} \mathcal{E}_{k,\gamma}(p),$$

and the function $\mathcal{E}_{k,\gamma}(p)$ is defined by (3.5).

Remark 4.1. We note that according to remarks 2.1 and 2.2, the spectrum of the two-particle Schrödinger operator $h_\mu(k)$, $k \in \mathbb{T}^3$, fills the following interval on the real axis:

$$\sigma_{\mathrm{ess}}(h_\mu(k)) = [\mathcal{E}_{\min,1}(k), \mathcal{E}_{\max,1}(k)].$$

Remark 4.2. We remark that when $k = (\pi, \pi, \pi) \in \mathbb{T}^3$ the essential spectrum of $h_{\mu,1}(k)$ degenerates to the set consisting of the unique point $\{\mathcal{E}_{\min}(k)\}$, and hence the essential spectrum of $h_{\mu,\gamma}(k)$ is not absolutely continuous for all $\gamma > 0$ and $k \in \mathbb{T}^3$.

We denote by $v^{\frac{1}{2}}$ the positive square root of the positive operator v.

For $z \leqslant \mathcal{E}_{\min,\gamma}(0)$ on the Hilbert space $L^2(\mathbb{T}^3)$, we shall consider the integral operator

$$G_{\mu,\gamma}(k, z) = \mu v^{\frac{1}{2}} (\mathcal{E}_{k,\gamma}(\cdot) - z)^{-1} v^{\frac{1}{2}}.$$

Remark 4.3. Clearly, the operator $h_{\mu,\gamma}(0)$ has an eigenvalue $z \leqslant \mathcal{E}_{\min,\gamma}(0) = \min_{p\in\mathbb{T}^3} \mathcal{E}_{0,\gamma}(q) = 0$, that is, $\mathrm{Ker}(h_{\mu,\gamma}(0) - zI) \neq 0$, if and only if the compact operator $G_{\mu,\gamma}(0, z)$ on $L^2(\mathbb{T}^3)$ has an eigenvalue 1 and there exists a function $\psi \in \mathrm{Ker}(G_{\mu,\gamma}(0, z) - I)$ such that the function f given by

$$f(p) = \frac{\big(v^{\frac{1}{2}}\psi\big)(p)}{\mathcal{E}_{0,\gamma}(p) - z}, \qquad \text{a.e.} \quad p \in \mathbb{T}^3, \tag{4.1}$$

belongs to $L^2(\mathbb{T}^3)$. In this case, $f \in \mathrm{Ker}(h_{\mu,\gamma}(0) - zI)$.

Moreover, if $z < \mathcal{E}_{\min,\gamma}(0)$, then

$$\dim \operatorname{Ker}(h_\mu(0) - zI) = \dim \operatorname{Ker}(G_{\mu,\gamma}(0, z) - I) \tag{4.2}$$

and

$$\operatorname{Ker}(h_{\mu,\gamma}(0) - zI) = \left\{ f \,|\, f(\cdot) = \frac{\left(v^{\frac{1}{2}}\psi\right)(\cdot)}{\mathcal{E}_{0,\gamma}(\cdot) - z},\ \psi \in \operatorname{Ker}(G_{\mu,\gamma}(0, z) - I) \right\}.$$

In the case of a threshold eigenvalue, $z = \mathcal{E}_{\min,\gamma}(0)$ the equality (4.2) may fail to hold. Hence, equality (4.2) should be replaced by the inequality

$$\dim \operatorname{Ker}(h_{\mu,\gamma}(0) - \mathcal{E}_{\min,\gamma}(0)I) \leqslant \dim \operatorname{Ker}(G_{\mu,\gamma}(0, z) - I).$$

Definition 4.4. *The operator $h_{\mu,\gamma}(0)$ is said to have a zero-energy resonance if the number 1 is an eigenvalue for the operator*

$$G_{\mu,\gamma}(0, 0) = \mu v^{\frac{1}{2}} (\mathcal{E}_{0,\gamma}(\cdot))^{-1} v^{\frac{1}{2}}, \tag{4.3}$$

and the associated eigenfunction ψ satisfies the condition $(v^{1/2}\psi)(0) \neq 0$. Without loss of generality, we can always normalize $(v^{1/2}\psi)(0)$ so that $(v^{1/2}\psi)(0) = 1$.

Remark 4.5. If ψ is an eigenfunction of $G_{\mu,\gamma}(0, 0)$ associated with eigenvalue 1, then the function

$$f(p) = (\varepsilon_{0,\gamma}(p))^{-1}(v^{1/2}\psi)(p)$$

is a simple solution (up to a constant factor) of the equation $h_{\mu,\gamma}(0)f = 0$.

Since

$$\varepsilon(p) = \tfrac{1}{2}|p|^2 + O(|p|^4) \qquad \text{as} \quad p \to 0,$$

in this case we have that the function

$$f(p) = (\varepsilon_{0,\gamma}(p))^{-1}(v^{1/2}\psi)(p)$$

belongs to $L^1(\mathbb{T}^3)\backslash L^2(\mathbb{T}^3)$, where $L^1(\mathbb{T}^3)$ is the Banach space of integrable functions.

Remark 4.6. We note that for any $\gamma > 0$ the number 1 is an eigenvalue of

$$G_\gamma = G_{\mu(\gamma),\gamma} = \mu(\gamma) v^{\frac{1}{2}} (\varepsilon(\cdot))^{-1} v^{\frac{1}{2}},$$

where

$$\mu(\gamma) = \mu_0(1 + \gamma) \qquad \text{and} \qquad \mu_0 = (2\pi)^3 \left(\int_{\mathbb{T}^3} (\varepsilon(p))^{-1}\, \mathrm{d}p \right)^{-1}. \tag{4.4}$$

Remark 4.6 allows us to introduce the following family of the two-particle and three-particle operators $\hbar_\gamma(k)$ and $H_\gamma(K)$ depending on $k \in \mathbb{T}^3$, $\gamma > 0$, and $K \in \mathbb{T}^3$, $\gamma > 0$, respectively:

$$\hbar_\gamma(k) \equiv h_{\mu(\gamma),\gamma}(k), \tag{4.5}$$

$$H_\gamma(K) \equiv H_{\mu(\gamma),\gamma}(K), \tag{4.6}$$

where $h_{\mu,\gamma}(k)$ and $H_{\mu,\gamma}(K)$ are defined by (3.3) and (3.7) respectively.

We recall that the main goal of the present paper is to prove either finiteness or infiniteness of the number of eigenvalues of the operators $H_\gamma(K)$ depending on the parameters $K \in \mathbb{T}^3$ (the three-particle quasi-momentum) and $\gamma > 0$ (ratio of the mass of the fermion to the boson).

For each $K \in \mathbb{T}^3$ and $\gamma > 0$, we set

$$E_{\min,\gamma}(K) = \min_{p,q\in\mathbb{T}^3} E(K,\gamma;p,q), \qquad E_{\max,\gamma}(K) = \max_{p,q\in\mathbb{T}^3} E(K,\gamma;p,q). \tag{4.7}$$

The main results of the present paper are as follows.

Theorem 4.7. *The two-particle operator $\hbar_\gamma(0)$ (defined by (4.5)) has a zero-energy resonance. For all $k \in \mathbb{T}_0^3$ the operator $\hbar_\gamma(k)$ has a unique positive eigenvalue $z_\gamma(k)$ below the bottom of the essential spectrum of $\hbar_\gamma(k)$, and $z_\gamma(k)$ is even and real analytic in $k \in \mathbb{T}_0^3$.*

We recall that $\tau_\gamma(K)$ denotes the bottom of the essential spectrum $\sigma_{\rm ess}(H_\gamma(K))$ and $N(K,\gamma;z)$ denotes the number of eigenvalues of $H_\gamma(K)$ below $z \leqslant \tau_\gamma(K)$.

Theorem 4.8. *The essential spectrum $\sigma_{\rm ess}(H_\gamma(K))$ of $H_\gamma(K)$, $K \in \mathbb{T}_0^3$, satisfies the equality*

$$\sigma_{\rm ess}(H_\gamma(K))) = [\tau_\gamma(K), E_{\max,\gamma}(K)]. \tag{4.8}$$

In the following theorems, we describe precisely the dependence of the number of eigenvalues of $H_\gamma(K)$ lying below the bottom $\tau_\gamma(K)$ of the essential spectrum $\sigma_{\rm ess}(H_\gamma(K))$ on the parameters $K \in \mathbb{T}^3$ and $\gamma > 0$.

Theorem 4.9. *(i) The operator $H_\gamma(K)$, $K \in \mathbb{T}_0^3$, $\gamma > 0$, has a finite number of eigenvalues lying below the bottom $\tau_\gamma(K)$ of the essential spectrum $\sigma_{\rm ess}(H_\gamma(K))$ and the function $N(K,\gamma;0)$ obeys the relation*

$$\lim_{|K|\to 0} \frac{N(K,\gamma;0)}{|\log|K||} = 2U(\gamma), \qquad (U(\gamma) > 0). \tag{4.9}$$

(ii) The operator $H_\gamma(0)$ has infinitely many eigenvalues lying below the bottom $\tau_\gamma(0)$ of the essential spectrum $\sigma_{\rm ess}(H_\gamma(0))$ and the function $N(0,\gamma;z)$ obeys the relation

$$\lim_{z\to 0-} \frac{N(0,\gamma;z)}{|\log|z||} = U(\gamma) > 0. \tag{4.10}$$

Remark 4.10. Clearly, the infinitude of the negative discrete spectrum of $H_\gamma(0)$ follows automatically from the positivity of $U(\gamma) > 0$.

Remark 4.11. We note that the bottom $\tau_\gamma(K)$, $K \in \mathbb{T}^3$, $\gamma > 0$, of the essential spectrum (see theorem 4.8) satisfies the conditions

$$\tau_\gamma(K) = \inf_{p\in\mathbb{T}^3} \{z_\gamma(K-p) + \varepsilon(p)\} > 0, \qquad K \in \mathbb{T}_0^3,$$

and

$$\tau_\gamma(0) = \inf_{p\in\mathbb{T}^3} \{z_\gamma(p) + \varepsilon(p)\} = 0.$$

Remark 4.12. Since the operator $V = I \otimes v$, defined by (3.8), is positive, we can conclude that $H_\gamma(K)$ has no eigenvalues above the essential spectrum $\sigma_{\rm ess}(H_\gamma(K))$. Consequently, according to theorem 4.9 the operator $H_\gamma(K)$ has either finitely many or infinitely many eigenvalues lying outside of $\sigma_{\rm ess}(H_\gamma(K))$.

5. Spectral properties of the two-particle operator $\hbar_\gamma(k)$

In this section, we study some spectral properties of the two-particle discrete Schrödinger operator $\hbar_\gamma(k)$, $k \in \mathbb{T}^3$, defined by (4.5).

Let $\mathbb{C}$ be the field of complex numbers. For any $k \in \mathbb{T}^3$ and $z \in \mathbb{C}\backslash[\mathcal{E}_{\min,\gamma}(k), \mathcal{E}_{\max,\gamma}(k)]$, we define a function (Fredholm's determinant associated with the operator $\hbar_\gamma(k)$)

$$\Delta_\gamma(k,z) = 1 - \frac{\mu(\gamma)}{(2\pi)^3}\int_{\mathbb{T}^3}(\mathcal{E}_{k,\gamma}(q)-z)^{-1}\,\mathrm{d}q, \tag{5.1}$$

where $\mu(\gamma)$ is defined in (4.4) and $\gamma > 0$.

Note that the function $\Delta_\gamma(k,z)$ is real analytic in $\mathbb{T}^3 \times (\mathbb{C}\backslash[\mathcal{E}_{\min,\gamma}(k); \mathcal{E}_{\min,\gamma}(k)])$.

The following lemma is a simple consequence of the Birman–Schwinger principle and Fredholm's theorem.

Lemma 5.1. *A number $z \in \mathbb{C}\backslash[\mathcal{E}_{\min,\gamma}(k), \mathcal{E}_{\max,\gamma}(k)]$ is an eigenvalue of the operator $\hbar_\gamma(k)$ if and only if $\Delta_\gamma(k,z) = 0$.*

The function $\mathcal{E}_{0,\gamma}(\cdot)$ has a unique non-degenerate minimum at $q = 0$ and hence by Lebesgue's dominated convergence theorem the following finite limit exists:

$$\lim_{z\to 0-}\Delta_\gamma(0,z) = \Delta_\gamma(0,0).$$

Lemma 5.2. *The operator $\hbar_\gamma(0)$ has a zero-energy resonance if and only if $\Delta_\gamma(0,0) = 0$.*

Proof. Lemma 5.2 can be proven in the same way as lemma 5.3 in [15]. □

Lemma 5.3. *For all $k \in \mathbb{T}^3_0$, the operator $\hbar_\gamma(k)$ is strictly positive.*

Proof. Since the operator $\hbar_\gamma(0)$ has a zero-energy resonance (remark 4.6), by lemma 5.1, we have

$$\Delta_\gamma(0,0) = 0$$

and hence the representation

$$\Delta_\gamma(k,0) = \frac{\mu_0}{(2\pi)^3}\int_{\mathbb{T}^3}\frac{\gamma(\varepsilon(k-q)-\varepsilon(q))}{\varepsilon(q)\mathcal{E}_{k,\gamma}(q)}\,\mathrm{d}q \tag{5.2}$$

holds. Making the change of variables $q = \frac{k}{2} - p$ in (5.2) and using the equality $\Delta_\gamma(k,0) = \Delta_\gamma(-k,0)$, it is easy to show that

$$\begin{aligned}\Delta_\gamma(k,0) &= \frac{\Delta_\gamma(k,0)+\Delta_\gamma(-k,0)}{2}\\ &= \frac{\mu_0}{2(2\pi)^3}\int_{\mathbb{T}^3}\gamma\left(\varepsilon\left(\frac{k}{2}+p\right)-\varepsilon\left(\frac{k}{2}-p\right)\right)^2 F(k,p)\,\mathrm{d}p,\end{aligned}$$

where

$$F(k,p) = \frac{\varepsilon\left(\frac{k}{2}+p\right)+\varepsilon\left(\frac{k}{2}-p\right)}{\varepsilon\left(\frac{k}{2}+p\right)\varepsilon\left(\frac{k}{2}-p\right)\mathcal{E}_{k,\gamma}\left(\frac{k}{2}+p\right)\mathcal{E}_{k,\gamma}\left(\frac{k}{2}-p\right)} > 0.$$

Thus for all $k \in \mathbb{T}^3_0$, the inequality

$$\Delta_\gamma(k,0) > 0.$$

is proven. Since for any $\gamma > 0$ and $k \in \mathbb{T}^3$ the function $\Delta_\gamma(k, \cdot)$ is monotone decreasing on $(-\infty, \mathcal{E}_{\min,\gamma}(k))$ for any $z < \mathcal{E}_{\min,\gamma}(k)$, the following inequalities hold:

$$\Delta_\gamma(k, z) > \Delta_\gamma(k, 0) > \Delta_\gamma(0, 0).$$

By lemma 5.1 the operator $\hbar_\gamma(k)$ have no negative eigenvalue, i.e. the operator $\hbar_\gamma(k)$ is positive. □

Proof of theorem 4.7. By definition (4.5) of the operator $\hbar_\gamma(0)$ for any $\gamma > 0$, it has a zero-energy resonance.

First we shall prove the existence of an eigenvalue of $\hbar_\gamma(k)$, $k \in \mathbb{T}_0^3$. The function $\mathcal{E}_{k,\gamma}(p)$ can be rewritten in the form

$$\mathcal{E}_{k,\gamma}(p) = 3(1+\gamma) - \sum_{j=1}^{3} \sqrt{1 + 2\gamma \cos k^{(j)} + \gamma^2} \cos(p_j - p_\gamma(k^{(j)})), \tag{5.3}$$

where

$$p_\gamma(k^{(j)}) = \arcsin \frac{\gamma \sin k^{(j)}}{\sqrt{1 + 2\gamma \cos k^{(j)} + \gamma^2}}, \qquad k^{(j)} \in (-\pi, \pi], \quad j = 1, 2, 3. \tag{5.4}$$

Taking into account (5.3), we have that the vector function

$$p_\gamma : \mathbb{T}^3 \to \mathbb{T}^3, \qquad p_\gamma(k) = p_\gamma(k^{(1)}, k^{(2)}, k^{(3)}) = (p_\gamma(k^{(1)}), p_\gamma(k^{(2)}), p_\gamma(k^{(3)})) \in \mathbb{T}^3$$

is odd and regular in $(-\pi, \pi)^3$ and

$$\min_{p \in \mathbb{T}^3} \mathcal{E}_{k,\gamma}(p) = \mathcal{E}_{k,\gamma}(p_\gamma(k)).$$

One has, as easily seen from the definition,

$$p_\gamma(k) = \frac{\gamma}{1+\gamma} k + O(|k|^3) \qquad \text{as} \quad k \to 0. \tag{5.5}$$

Moreover from (5.3), it follows that

$$\mathcal{E}_{\min,\gamma}(k) = \mathcal{E}_{k,\gamma}(p_\gamma(k)) = 3(1+\gamma) - \sum_{j=1}^{3} \sqrt{1 + 2\gamma \cos k^{(j)} + \gamma^2}. \tag{5.6}$$

In the case $\gamma \neq 1$ and $k \in \mathbb{T}^3$, the point $p = p_\gamma(k)$ is the non-degenerate minimum of the function $\mathcal{E}_{k,\gamma}(p)$ and hence the function $(\mathcal{E}_{k,\gamma}(p) - \mathcal{E}_{\min,\gamma}(k))^{-1}$ is integrable. We define $\Delta_\gamma(k, \mathcal{E}_{\min,\gamma}(k))$ by

$$\Delta_\gamma(k, \mathcal{E}_{\min,\gamma}(k)) = 1 - \frac{\mu(\gamma)}{(2\pi)^3} \int_{\mathbb{T}^3} (\mathcal{E}_{k,\gamma}(q) - \mathcal{E}_{\min,\gamma}(k))^{-1} \, \mathrm{d}q.$$

By the dominated convergence theorem, we have

$$\lim_{z \to \mathcal{E}_{\min,\gamma}(k) - 0} \Delta_\gamma(k, z) = \Delta_\gamma(k, \mathcal{E}_{\min,\gamma}(k)).$$

In the case $\gamma = 1$ and $k \in \mathbb{T}^3 \backslash (-\pi, \pi)^3$, we have

$$\Delta_\gamma(k, \mathcal{E}_{\min,\gamma}(k)) = \frac{\mu(\gamma)}{(2\pi)^3} \int_{\mathbb{T}^3} (\mathcal{E}_{k,\gamma}(q) - \mathcal{E}_{\min,\gamma}(k))^{-1} \, \mathrm{d}q = -\infty.$$

From the representations (5.3) and (5.6), it follows that for all $k \neq 0$, $q \neq 0$ the inequality

$$\mathcal{E}_{k,\gamma}(q + p_\gamma(k)) - \mathcal{E}_{\min,\gamma}(k) < \mathcal{E}_{0,\gamma}(q)$$

holds. Hence by virtue of lemma 5.2, we obtain the following inequality:

$$\Delta_\gamma(k, \mathcal{E}_{\min,\gamma}(k)) < \Delta_\gamma(0, 0) = 0, \qquad k \in \mathbb{T}_0^3. \tag{5.7}$$

For each $\gamma > 0$ and $k \in \mathbb{T}^3$, the function $\Delta_\gamma(k, \cdot)$ is continuous monotone decreasing on $(-\infty, \mathcal{E}_{\min,\gamma}(k))$ and $\Delta_\gamma(k, z) \to 1$ as $z \to -\infty$. Therefore by virtue of (5.7), there is a unique number $z_\gamma(k) \in (-\infty, \mathcal{E}_{\min,\gamma}(k))$ such that $\Delta_\gamma(k, z_\gamma(k)) = 0$. By lemma 5.1 for any nonzero $k \in \mathbb{T}^3_0$, the operator $\hbar_\gamma(k)$ has a unique eigenvalue below $\mathcal{E}_{\min,\gamma}(k)$.

Let us prove the positivity of the eigenvalue $z_\gamma(k)$, $k \in \mathbb{T}^3_0$. For any $\gamma > 0$ and $k \in \mathbb{T}^3_0$ by lemma 5.3 we have $\Delta_\gamma(k, 0) > 0$ and the function $\Delta_\gamma(k, \cdot)$ is monotone decreasing and hence the inequalities

$$\Delta_\gamma(k, 0) > \Delta_\gamma(k, z_\gamma(k)) = 0 > \Delta_\gamma(k, \mathcal{E}_{\min,\gamma}(k)), \qquad k \in \mathbb{T}^3_0,$$

hold. Therefore, the eigenvalue $z_\gamma(k)$ of the operator $\hbar_\gamma(k)$ belongs to $(0, \mathcal{E}_{\min,\gamma}(k))$.

Since $z_\gamma(k)$ is a solution of the equation $\Delta_\gamma(k, z) = 0$ and for any $\gamma > 0, z \in (-\infty, \mathcal{E}_{\min,\gamma}(k))$ the equality $\Delta_\gamma(-k, z) = \Delta_\gamma(k, z), k \in \mathbb{T}^3$ holds, the function $z_\gamma(\cdot)$ is real analytic and even in $\mathbb{T}^3_0$. □

Now we derive an asymptotics for Fredholm's determinant $\Delta(0, z)$ as $z \to 0$, which plays a crucial role in the proof of the main results (asymptotics (4.10)). In particular, we show that the function $\Delta(0, \mathcal{E}_{\min,\gamma}(k) - \omega^2)$ is differentiable in ω at $\omega^2 = \mathcal{E}_{\min,\gamma}(k)$.

We note that in the continuous case (see [8, 10]) for this aim, the resolvent expansion obtained in [29] has been used.

Lemma 5.4. *For any $\gamma > 0$ and $k \in \mathbb{T}^3$ and $z \leqslant \mathcal{E}_{\min,\gamma}(k)$, the following decomposition holds:*

$$\Delta_\gamma(k, z) = \frac{\mu(\gamma)\gamma^{3/2}}{\sqrt{2}\pi(1+\gamma)^{3/2}}[\mathcal{E}_{\min,\gamma}(k) - z]^{\frac{1}{2}} + \Delta^{(20)}_\gamma(\mathcal{E}_{\min,\gamma}(k) - z) + \Delta^{(02)}_\gamma(k, z), \tag{5.8}$$

where $[\mathcal{E}_{\min,\gamma}(k) - z]^{\frac{1}{2}} > 0$ for $\mathcal{E}_{\min,\gamma}(k) - z > 0$ and $\Delta^{(20)}_\gamma(\mathcal{E}_{\min,\gamma}(k) - z) = O(\mathcal{E}_{\min,\gamma}(k) - z)$, as $z \to \mathcal{E}_{\min,\gamma}(k)$ and $\Delta^{(02)}_\gamma(k, z) = O(|k|^2)$, uniformly in $z \leqslant \mathcal{E}_{\min,\gamma}(k)$, as $k \to 0$.

Proof. Let

$$E_\gamma(k, p) = \mathcal{E}_{k,\gamma}(p + p_\gamma(k)) - \mathcal{E}_{\min,\gamma}(k).$$

Then using (5.3), we conclude

$$E_\gamma(k, p) = \sum_{j=1}^{3} \sqrt{1 + 2\gamma\cos k_j + \gamma^2}(1 - \cos p_j).$$

We define the function $\widetilde{\Delta}_\gamma(k, \omega)$ on $\mathbb{T}^3 \times \mathbb{C}_+$ by $\widetilde{\Delta}_\gamma(k, \omega) = \Delta_\gamma(k, \mathcal{E}_{\min,\gamma}(k) - \omega^2)$, where $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re} z > 0\}$. The function $\widetilde{\Delta}_\gamma(\cdot, \omega)$ is even in $k \in \mathbb{T}^3$. The function $\widetilde{\Delta}_\gamma(k, \omega)$ can be represented in the following way:

$$\begin{aligned}\widetilde{\Delta}_\gamma(k, \omega) &= 1 - \mu(\gamma)(2\pi)^{-3}\int_{\mathbb{T}^3}\frac{\mathrm{d}p}{E_\gamma(k, p) + \omega^2}\\ &= 1 - \mu(\gamma)(2\pi)^{-3}\int_{\mathbb{T}^3}\frac{\mathrm{d}p}{\sum_{j=1}^{3}\sqrt{1 + 2\gamma\cos k_j + \gamma^2}(1 - \cos p_j) + \omega^2}.\end{aligned}$$

Let $V_\delta(0)$ be a complex δ-neighborhood of the point $\omega = 0 \in \mathbb{C}$. Denote by $\Delta^*_\gamma(k, \omega)$ the analytic continuation of the function $\widetilde{\Delta}_\gamma(k, \omega)$ to the region $\mathbb{T}^3 \times (\mathbb{C}_+ \cup V_\delta(0))$ (see [30]). A Taylor series expansion gives

$$\Delta^*_\gamma(k, \omega) = \Delta^*_\gamma(0, \omega) + \widetilde{\Delta}^{(02)}_\gamma(k, \omega),$$

where $\widetilde{\Delta}_\gamma^{(02)}(k,\omega) = O(|k|^2)$, uniformly in $\omega \in \mathbb{C}_+$ as $k \to 0$. The Taylor series expansion gives

$$\Delta_\gamma^*(0,w) = \Delta_\gamma^*(0,0)w + \tilde{\Delta}_\gamma^{(02)}(0,w)w^2,$$

where $\tilde{\Delta}_\gamma^{(02)}(0,w) = O(1)$ as $w \to 0$. Then a simple computation shows that

$$\frac{\partial \Delta_\gamma^*(0,0)}{\partial \omega} = \Delta_\gamma^*(0,0) = \frac{\mu(\gamma)\gamma^{3/2}}{\sqrt{2}\pi(1+\gamma)^{3/2}} \neq 0. \tag{5.9}$$

The equality $\mathcal{E}_{\min,\gamma}(k) - z = \omega^2$ yields the proof of lemma. □

Lemma 5.5. *Let $w_\gamma(k)$ be the simple solution of the equation $\Delta_\gamma^*(k,\omega) = 0$, $(k,w) \in \mathbb{T}^3 \times [0,+\infty)$. Then $w_\gamma(k) = O(|k|^2)$ as $k \to 0$.*

Proof. By theorem 4.7 and lemmas 5.1, 5.2, the equation $\Delta_\gamma^*(k,w) = 0$ has a simple real-analytic solution $w_\gamma(k)$, $k \in \mathbb{T}^3$. Taking into account that the function $\Delta_\gamma^*(k,w)$ is even in $k \in \mathbb{T}_0^3$ and $w_\gamma(0) = 0$, we have that $w_\gamma(k)$ is even and $w_\gamma(k) = O(|k|^2)$ as $k \to 0$. Therefore, the function $z_\gamma(k) = \mathcal{E}_{\min,\gamma}(k) - w_\gamma^2(k)$ is real analytic in $\mathbb{T}^3$ and $\mathcal{E}_{\min,\gamma}(k) - z_\gamma(k) = w_\gamma^2(k) = O(|k|^4)$. □

Lemma 5.6. *For any $k \in \mathbb{T}_0^3$ there exists a number $\delta(k) > 0$ such that, for all $z \in V_{\delta(k)}(z_\gamma(k))$, where $V_{\delta(k)}(z_\gamma(k))$ is the $\delta(k)$-neighborhood of the point $z_\gamma(k)$, the following representation holds:*

$$\Delta_\gamma(k,z) = C_1(k)(z - z_\gamma(k))\hat{\Delta}_\gamma(k,z).$$

Here $C_1(k) \neq 0$ and $\hat{\Delta}_\gamma(k,z)$ is regular in $V_{\delta(k)}(z_\gamma(k))$ and $\hat{\Delta}_\gamma(k, z_\gamma(k)) \neq 0$.

Proof. The number $z_\gamma(k) < \mathcal{E}_{\min,\gamma}(k)$, $k \neq 0$, is a simple solution of $\Delta_\gamma(k,z) = 0$ and the function $\Delta_\gamma(k,z)$ is regular in $\mathbb{C}\backslash[\mathcal{E}_{\min,\gamma}(k); \mathcal{E}_{\max,\gamma}(k)]$. Hence for some $\delta(k) > 0$, the function Δ can be expanded as

$$\Delta_\gamma(k,z) = \sum_{n=1}^{\infty} C_n(k)(z - z_\gamma(k))^n, \qquad z \in V_{\delta(k)}(z_\gamma(k)),$$

where

$$C_1(k) = \frac{\mu(\gamma)\gamma^{3/2}}{\sqrt{2}\pi(1+\gamma)^{3/2}} \frac{1}{2\sqrt{\mathcal{E}_{\min,\gamma}(k) - z_\gamma(k)}} \neq 0, \qquad k \neq 0,$$

and

$$\hat{\Delta}_\gamma(k,z) = \sum_{n=1}^{\infty} \frac{C_n(k)}{C_1(k)}(z - z(k))^{n-1}$$

Clearly, $\hat{\Delta}_\gamma(k,z)$ is regular in $V_{\delta(k)}(z_\gamma(k))$. Since $z_\gamma(k)$, $k \neq 0$, is the unique simple solution of the equation $\Delta_\gamma(k,z) = 0$, $k \in \mathbb{T}^3$, $z < \mathcal{E}_{\min,\gamma}(k)$, we have the inequality $\hat{\Delta}_\gamma(k, z_\gamma(k)) \neq 0$. □

6. The channel operators and the essential spectrum of $H_\gamma(K)$, $K \in \mathbb{T}^3$

Recall that we consider a three-particle system consisting of two identical fermions and one boson. The fermions and the boson interact via a zero-range pair attractive potential.

Therefore, we have only one non-trivial channel operator $H_\gamma^{\rm ch}(K)$, $K\in\mathbb{T}^3$ acting on $L^2((\mathbb{T}^3)^2)$ as

$$\big(H_\gamma^{\rm ch}(K)f\big)(p,q)=E(K,\gamma;p,q)f(p,q)-\frac{\mu(\gamma)}{(2\pi)^3}\int_{\mathbb{T}^3}f(p,q')\,{\rm d}q',\qquad(6.1)$$

where $E(K,\gamma;p,q)$ is defined by (3.9).

Since the operator $H_\gamma^{\rm ch}(K)$ commutes with the group $\{U_s^{(2)},s\in\mathbb{Z}^3\}$ of the unitary operators

$$\big(U_s^{(2)}f\big)(p,q)=\exp\{-{\rm i}(s,p)\}f(p,q),\qquad f\in L^{2,a}((\mathbb{T}^3)^2),$$

the decomposition of the Hilbert space $L^2((\mathbb{T}^3)^2)$ into the direct integral

$$L^2((\mathbb{T}^3)^2)=\int_{\mathbb{T}^3}\oplus L^2(\mathbb{T}^3)\,{\rm d}p$$

yields the decomposition

$$H_\gamma^{\rm ch}(K)=\int_{\mathbb{T}^3}\oplus H_\gamma^{\rm ch}(K,p)\,{\rm d}p.$$

The fiber operator $H_\gamma^{\rm ch}(K,p)$ acts in $L^2(\mathbb{T}^3)$ by

$$H_\gamma^{\rm ch}(K,p)=\hbar_\gamma(K-p)+\varepsilon(p)I_{L^2(\mathbb{T}^3)},\qquad(6.2)$$

where $I_{L^2(\mathbb{T}^3)}$ is the identity operator on $L^2(\mathbb{T}^3)$ and $\hbar_\gamma(k)$ is the two-particle operator defined by (4.5).

The representation (6.2) of the operator $H_\gamma^{\rm ch}(K,p)$ implies for the spectrum $\sigma\big(H_\gamma^{\rm ch}(K,p)\big)$ the equality

$$\sigma\big(H_\gamma^{\rm ch}(K,p)\big)=\{z_\gamma(K-p)+\varepsilon(p)\}\cup[\mathcal{E}_{\min,\gamma}(K-p)+\varepsilon(p),\mathcal{E}_{\max,\gamma}(K-p)+\varepsilon(p)].\qquad(6.3)$$

Remark 6.1. We note that the number $\tau_\gamma(0)=\tau_\gamma(0,0)=z_\gamma(0)+\varepsilon(0)=0$ is a zero-energy resonance for $H_\gamma^{\rm ch}(0,0)$ and the positive number $\tau_\gamma(K,p)=z_\gamma(K-p)+\varepsilon(p)>0$ is an eigenvalue for $H_\gamma^{\rm ch}(K,p)$, $K\in\mathbb{T}_0^3$, $p\in\mathbb{T}^3$.

Lemma 6.2. *The operator $H_\gamma^{\rm ch}(K)$, $K\in\mathbb{T}^3$, is positive and the equality*

$$\sigma(H_\gamma^{\rm ch}(K))=[\tau_\gamma(K),E_{\max,\gamma}(K)]\qquad(6.4)$$

holds, where $E_{\max,\gamma}(K)$ is the maximum value of the function $E(K,\gamma;p,q)$ defined by (3.9).

Proof. The theorem (see, e.g., [26]) on the spectrum of decomposable operators and the structure (6.3) of the spectrum of $H_\gamma^{\rm ch}(K,p)$ gives the result. □

Lemma 6.3. *For any $K\in\mathbb{T}_0^3$, the following inequality holds:*

$$\tau_\gamma(K)<E_{\min,\gamma}(K).$$

Proof. For any $K\in\mathbb{T}_0^3$, by virtue of theorem 4.7 the inequality

$$z_\gamma(K-p)+\varepsilon(p)<\mathcal{E}_{\min,\gamma}(K-p)+\varepsilon(p),\qquad p\in\mathbb{T}^3,$$

holds. The result then follows by applying the theorem (see, e.g., [26]) on the spectrum of decomposable operators. □

Theorem 6.4. *For the essential spectrum $\sigma_{\rm ess}(H_\gamma(K))$ of $H_\gamma(K)$, the equality*

$$\sigma(H_\gamma^{\rm ch}(K))=\sigma_{\rm ess}(H_\gamma(K))$$

holds.

The proof of theorem 6.4 is similar to the proof of theorem 4.3 in [15].

Proof of theorem 4.8. Theorem 4.8 follows from lemma 6.2 and theorem 6.4. □

7. The Birman–Schwinger principle for $H_\gamma(K)$

In this section, we prove an extension of the well-known Birman–Schwinger principle for the two-particle Schrödinger operators, to the case of the three-particle Schrödinger operator $H_\gamma(K)$, $K \in \mathbb{T}^3$(the Birman–Schwinger principle for the continuous three-particle Schrödinger operators can be found in [8, 10]).

By lemma 6.2 for all $K \in \mathbb{T}^3$ the operator $H_\gamma^{\mathrm{ch}}(K)$ has no spectrum below the bottom $\tau_\gamma(K)$ and hence for any $z < \tau_\gamma(K)$ its resolvent $R_\gamma^{\mathrm{ch}}(K,z)$ exists and is positive.

Let $W_\gamma(K,z)$, $K \in \mathbb{T}^3$, $z < \tau_\gamma(K)$, be the operators on $L^2((\mathbb{T}^3)^2)$ defined as

$$W_\gamma(K,z) = I + V^{\frac12} R_\gamma^{\mathrm{ch}}(K,z) V^{\frac12},$$

where $V^{\frac12} = I \otimes v^{\frac12}$ and $v^{\frac12}$ is the square root of v defined by (3.6).

One checks that

$$W_\gamma(K,z) = \big(I - V^{\frac12} R_\gamma^0(K,z) V^{\frac12}\big)^{-1},$$

where $R_\gamma^0(K,z)$ is the resolvent of the operator $H_\gamma^0(K)$. Clearly, $W_\gamma(K,z) \geqslant I$. Therefore for all $K \in \mathbb{T}^3$, $z < \tau_\gamma(K)$ the operators $V^{\frac12} R_\gamma^0(K,z) V^{\frac12}$ are positive.

Denote by

$$\mathbf{T}_\gamma(K,z), \qquad K \in \mathbb{T}^3, \qquad z < \tau_\gamma(K)$$

the operator in $L^{2,a}((\mathbb{T}^3)^2)$ defined by

$$\mathbf{T}_\gamma(K,z) = 2 W_\gamma^{\frac12}(K,z) V^{\frac12} R_\gamma^0(K,z) V^{\frac12} W_\gamma^{\frac12}(K,z). \tag{7.1}$$

For any bounded self-adjoint operator A acting in the Hilbert space $\mathcal{H}$ not having any essential spectrum on the right of the point z, we denote by $\mathcal{H}_A(z)$ the subspace such that $(Af, f) > z(f, f)$ for any $f \in \mathcal{H}_A(z)$ and set $n(z, A) = \sup_{\mathcal{H}_A(z)} \dim \mathcal{H}_A(z)$. By the definition of $N(K,\gamma;z)$, we have

$$N(K,\gamma;z) = n(-z, -H_\gamma(K)), \qquad -z > -\tau_\gamma(K).$$

Remark 7.1. By theorem 4.8 for any $K \in \mathbb{T}^3$ the operators $H_\gamma(K)$ have no essential spectrum below $z < \tau_\gamma(K)$ and hence the operators $-H_\gamma(K)$ have no essential spectrum above $-z > -\tau_\gamma(K)$.

Theorem 7.2. *For $z < \tau_\gamma(K)$, the operator $\mathbf{T}_\gamma(K,z)$ is compact and continuous in z and*

$$N(K,\gamma;z) = n(1, \mathbf{T}(K,z)).$$

Proof. We first verify the equality

$$N(K,\gamma;z) = n\big(1, 2\big(R_\gamma^0(K,z)\big)^{\frac12} V \big(R_\gamma^0(K,z)\big)^{\frac12}\big). \tag{7.2}$$

Assume that $u \in \mathcal{H}_{-H_\gamma(K)}(-z)$, that is, $((H_\gamma^0(K) - z)u, u) < 2(Vu, u)$. Then

$$(y,y) < 2\big(\big(R_\gamma^0(K,z)\big)^{\frac12} V \big(R_\gamma^0(K,z)\big)^{\frac12} y, y\big), \qquad y = \big(H_\gamma^0(K) - z\big)^{\frac12} u.$$

Thus, $N(K,\gamma;z) \leqslant n\big(1, 2\big(R_\gamma^0(K,z)\big)^{\frac12} V \big(R_\gamma^0(K,z)\big)^{\frac12}\big)$. Reversing the argument we get the opposite inequality, which proves (7.2). Any nonzero eigenvalue of $\big(R_\gamma^0(K,z)\big)^{\frac12} V^{\frac12}$ is an eigenvalue for $V^{\frac12}\big(R_\gamma^0(K,z)\big)^{\frac12}$ as well, of the same algebraic and geometric multiplicities. Therefore, we get

$$n\big(1, 2\big(R_\gamma^0(K,z)\big)^{\frac12} V \big(R_\gamma^0(K,z)\big)^{\frac12}\big) = n\big(1, 2V^{\frac12} R_0(K,z) V^{\frac12}\big).$$

Let us check that

$$n\big(1, 2\big(R_\gamma^0(K,z)\big)^{\frac12} V \big(R_\gamma^0(K,z)\big)^{\frac12}\big) = n\big(1, \mathbf{T}_\gamma(K,z)\big).$$

We shall show that for any $u \in \mathcal{H}_{2(R_\gamma^0(K,z))^{\frac12} V (R_\gamma^0(K,z))^{\frac12}}(1)$, there exists $y \in \mathcal{H}_{\mathbf{T}_\gamma(K,z)}(1)$ such that $(y, y) < (\mathbf{T}_\gamma(K,z)y, y)$. Let $u \in \mathcal{H}_{2(R_\gamma^0(K,z))^{\frac12} V (R_\gamma^0(K,z))^{\frac12}}(1)$, that is,

$$(u, u) < 2\big(V^{\frac12} R_\gamma^0(K,z) V^{\frac12} u, u\big)$$

and hence

$$\big((I - V^{\frac12} R_\gamma^0(K,z) V^{\frac12})u, u\big) < \big(V^{\frac12} R_\gamma^0(K,z) V^{\frac12} u, u\big). \tag{7.3}$$

Setting $y = \big(I - V^{\frac12} R_\gamma^0(K,z) V^{\frac12}\big)^{\frac12} u$, we have

$$(y, y) < 2\big(W_\gamma^{\frac12}(K,z) V^{\frac12} R_\gamma^0(K,z) V^{\frac12} W_\gamma^{\frac12}(K,z) y, y\big),$$

that is, $(y, y) \leqslant (\mathbf{T}_\gamma(K,z)y, y)$. Thus, $n\big(1, 2\big(R_\gamma^0(K,z)\big)^{\frac12} V \big(R_\gamma^0(K,z)\big)^{\frac12}\big) \leqslant n(1, \mathbf{T}(K,z))$.

In the same way, one checks that $n(1, \mathbf{T}_\gamma(K,z)) \leqslant n\big(1, 2\big(R_\gamma^0(K,z)\big)^{\frac12} V \big(R_\gamma^0(K,z)\big)^{\frac12}\big)$. □

Remark 7.3. On the left-hand side of (7.3) the operator $V^{\frac12} R_\gamma^0(K,z) V^{\frac12}$ is a partial integral operator, since the operator

$$V^{\frac12} f(k_\alpha, k_3) = V_\alpha^{\frac12} f(k_\alpha, k_3) = (I \otimes v^{\frac12}) f(k_\alpha, k_3)$$

is written in the coordinates (k_α, k_3), i.e. it is an integral operator with respect to k_3. The right-hand side of (7.3) can be written as $V_\alpha^{\frac12} R_\gamma^0(K,z) V_3^{\frac12}$, where the operator $V = V_\alpha$ is written in the coordinates (k_α, k_3), that is, it is an integral operator with respect to k_3. But the operator $V = V_3$ is written in the coordinates (k_3, k_α), that is, it is an integral operator with respect to k_α and hence the operator $V^{\frac12} R_\gamma^0(K,z) V^{\frac12}$ on the right-hand side of (7.3) is an integral operator in all variables.

8. The finiteness of the number of eigenvalues of $H_\gamma(K)$, $K \in \mathbb{T}_0^3$

Now we are going to proof the finiteness of $N(K, \gamma; \tau_\gamma(K))$ for $K \in \mathbb{T}_0^3$. First we shall prove that the operator $T_\gamma(K, \tau_\gamma(K))$, $K \in \mathbb{T}_0^3$, belongs to the Hilbert–Schmidt class.

The point $p = 0$ is the non-degenerate minimum of the function $\varepsilon(p)$ and the minimum for $z_\gamma(p)$, and hence $p = 0$ is the non-degenerate minimum of $Z_\gamma(0, p)$ defined by

$$Z_\gamma(0, p) := \varepsilon(p) + z_\gamma(p).$$

One can conclude that the minimum point $p_\gamma^Z(K) \in \mathbb{T}^3$ of the function $Z_\gamma(K, p) = \varepsilon(p) + z_\gamma(K - p)$, $K \in \mathbb{T}_0^3$ is non-degenerate, i.e.

$$B(K) = \left[\frac{\partial^2 Z_\gamma}{\partial p^{(i)} \partial p^{(j)}}\big(K, p_\gamma^Z(K)\big)\right]_{i,j=1}^3 > 0.$$

Hence, the asymptotics

$$Z_\gamma(K, p) = \tau_\gamma(K) + (B(K)(p - p_\gamma^Z(K)), p - p_\gamma^Z(K)) + o(|p - p_\gamma^Z(K)|^2) \quad \text{as} \quad |p - p_\gamma^Z(K)| \to 0 \tag{8.1}$$

holds, where $\tau_\gamma(K) = Z_\gamma\big(K, p_\gamma^Z(K)\big)$. From lemma 5.6 we conclude that for all $K \in \mathbb{T}_0^3$, $p \in U_{\delta(K)}\big(p_\gamma^Z(K)\big)$ the equality

$$\Delta_\gamma(K, p, \tau_\gamma(K)) = (Z_\gamma(K, p) - \tau_\gamma(K))\hat{\Delta}_\gamma(K, p, \tau_\gamma(K)) \tag{8.2}$$

holds, where $\hat{\Delta}_\gamma\big(K, p_\gamma^Z(K), \tau_\gamma(K)\big) \neq 0$. Putting (8.1) into (8.2), we get the following.

Lemma 8.1. *For any $K \in \mathbb{T}_0^3$ there are positive nonzero constants c and C depending on K and $U_{\delta(K)}(p_\gamma^Z(K))$ such that for all $p \in U_{\delta(K)}(p_\gamma^Z(K))$ the following inequalities hold:*

$$c\big|p - p_\gamma^Z(K)\big|^2 \leqslant \Delta_\gamma(K, p, \tau_\gamma(K)) \leqslant C\big|p - p_\gamma^Z(K)\big|^2. \tag{8.3}$$

Remark 8.2. Let the kernel function $v(\cdot)$ of the interaction operator v and the dispersion relation ε be real-analytic functions on the three-dimensional torus $\mathbb{T}^3$. In this case, the minimum (critical) values of the function $Z_\gamma(K, \cdot)$ may degenerate only at a finite number of manifolds $\aleph_n \in \mathbb{T}^3$, $n = 1, 2, \ldots, N$, of co-dimension 1.

Lemma 8.3. *For any $K \in \mathbb{T}_0^3$, the operator $T_\gamma(K, \tau_\gamma(K))$ belongs to the Hilbert–Schmidt class.*

Proof. By lemma 6.3 for the bottom $\tau_\gamma(K)$ of the essential spectrum, the inequality

$$\tau_\gamma(K) < E_{\min,\gamma}(K), \qquad K \in \mathbb{T}_0^3, \tag{8.4}$$

holds. Since the operator $\hbar_\gamma(0)$ has a zero-energy resonance, the operator $\hbar_\gamma(k)$, $k \in \mathbb{T}^3$, $k \neq 0$, has a unique eigenvalue $z_\gamma(k) < \mathcal{E}_{\min,\gamma}(k)$ (theorem 4.7).

The function $Z_\gamma(K, p)$ has a unique minimum and $\tau_\gamma(K) = \min_{p\in\mathbb{T}^3} Z_\gamma(K, p)$. Hence for all $p \in \mathbb{T}^3 \backslash U_\delta\big(p_\gamma^Z(K)\big)$, we obtain

$$\Delta_\gamma(K, p, \tau_\gamma(K)) \geqslant C(K) > 0. \tag{8.5}$$

According to (4.7) and (8.4) for all $p, q \in \mathbb{T}^3$ and $K \in \mathbb{T}_0^3$, inequality

$$E(K, \gamma; p, q) - \tau_\gamma(K) \geqslant E_{\min,\gamma}(K) - \tau_\gamma(K) > 0 \tag{8.6}$$

holds. Using (8.3), (8.5) and taking into account (8.6), we can make certain that for each $K \in \mathbb{T}_0^3$ and all $p \in U_\delta(p_\gamma^Z(K))$, $q \in U_\delta(p_\gamma^Z(K))$ the modules of the kernels $T_\gamma(K, \tau_\gamma(K); p, q)$ of the integral operators $T_\gamma(K, \tau_\gamma(K))$, $K \in \mathbb{T}_0^3$, can be estimated by

$$\frac{C_0(K)}{\big|p - p_\gamma^Z(K)\big|\big|q - p_\gamma^Z(K)\big|} + C_1(K),$$

where $C_0(K)$ and $C_1(K)$ are some constants. From this, we conclude that

$$T_\gamma(K, \tau_\gamma(K)), \qquad K \in \mathbb{T}_0^3$$

are Hilbert–Schmidt operators. □

Now we shall prove the finiteness of $N(K, \gamma; \tau_\gamma(K))$, $K \in \mathbb{T}_0^3$ (part (i) of theorem 4.9).

Let $T_\gamma(K, z)$, $K \in \mathbb{T}^3$, $z \leqslant \tau_\gamma(K)$, be the self-adjoint operator defined in $L^2(\mathbb{T}^3)$ by

$$(T_\gamma(K, z) f)(p) = \frac{\mu(\gamma)}{(2\pi)^3} \int_{\mathbb{T}^3} \frac{\Delta_\gamma^{-\frac{1}{2}}(K, p, z)\Delta_\gamma^{-\frac{1}{2}}(K, q, z)}{E(K, \gamma; p, q) - z} f(q)\,\mathrm{d}q. \tag{8.7}$$

Lemma 8.4. *The following equality holds:*

$$n(1, \mathbf{T}_\gamma(K, z)) = n(1, T_\gamma(K, z)).$$

Proof. Let $\Psi : L_2((\mathbb{T}^3)^2) \to L_2(\mathbb{T}^3)$ be the operator given by

$$(\Psi f)(p) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{T}^3} f(p, q)\, \mathrm{d}q$$

and let Ψ^* be its adjoint. One can easily check that the following equalities hold:

$$\Psi f = (2\pi)^{\frac{3}{2}} V^{\frac{1}{2}} f \qquad \text{and} \qquad V^{\frac{1}{2}} W_\gamma^{\frac{1}{2}} f = \Delta_\gamma^{-\frac{1}{2}}(K, p, z) V^{\frac{1}{2}} f, \quad f \in L_2((\mathbb{T}^3)^2). \tag{8.8}$$

These equalities imply the equality $\mathbf{T}_\gamma(K, z) = \Psi^* T_\gamma(K, z)\Psi$.

Since any nonzero eigenvalue of $\Psi^* T_\gamma(K, z)\Psi$ is an eigenvalue of $\Psi\Psi^* T_\gamma(K, z)$ as well, with the same algebraic and geometric multiplicities, and $\Psi\Psi^* = I_{L^2(\mathbb{T}^3)}$, we have

$$n(1, \mathbf{T}_\gamma(K, z)) = n(1, T_\gamma(K, z)).$$

□

Theorem 8.5. *For the number $N(K, \gamma; \tau_\gamma(K))$, $K \in \mathbb{T}_0^3$, the relation*

$$N(K, \gamma; \tau_\gamma(K)) \leqslant \lim_{\nu \to 0} n(1 - \nu, T_\gamma(K, \tau_\gamma(K))$$

holds.

Proof. By theorem 7.2 and lemma 8.4, we have

$$N(K, \gamma; z) = n(1, T_\gamma(K, z)) \qquad \text{as} \quad z < \tau_\gamma(K)$$

and by lemma 8.3 for any $\nu \in [0, 1)$ the number $n(1 - \nu, T_\gamma(K, \tau_\gamma(K)))$, $K \in \mathbb{T}_0^3$, is finite. Then according to the Weyl inequality

$$n(\lambda_1 + \lambda_2, A_1 + A_2) \leqslant n(\lambda_1, A_1) + n(\lambda_2, A_2),$$

for all $z < \tau_\gamma(K)$ and $\nu \in (0, 1)$ we have

$$N(K, \gamma; z) = n(1, T_\gamma(K, z)) \leqslant n(1 - \nu, T_\gamma(K, \tau_\gamma(K))) + n(\nu, T_\gamma(K, z) - T_\gamma(K, \tau_\gamma(K))).$$

Since $T_\gamma(K, z)$ is continuous from the left up to $z = \tau_\gamma(K)$, $K \in \mathbb{T}_0^3$, we obtain

$$\lim_{z \to \tau_\gamma(K)} N(K, \gamma; z) = N(K, \gamma; \tau_\gamma(K))) \leqslant n(1 - \nu, T_\gamma(K, \tau_\gamma(K))) \qquad \text{for all} \quad \nu \in (0, 1)$$

and so

$$T_\gamma(K, \tau_\gamma(K)) \leqslant \lim_{\nu \to 0} n(1 - \nu, T_\gamma(K, \tau_\gamma(K))), \qquad K \in \mathbb{T}_0^3.$$

□

9. Asymptotics for the number of eigenvalues of $H_\gamma(K)$

In this section, we shall prove theorem 4.9.

Theorem 9.1. *The following equality holds:*

$$\lim_{\frac{|K|^2}{M_\gamma} + |z| \to 0} \frac{n(1, \mathbf{T}_\gamma(K, z))}{\left|\log\left(\frac{|K|^2}{M_\gamma} + |z|\right)\right|} = U(\gamma). \tag{9.1}$$

Theorem 4.9 will be deduced by a perturbation argument based on lemma 4.7, which has been proven in [8]. For completeness, we here reproduce the lemma.

Lemma 9.2. *Let $A(z) = A_0(z) + A_1(z)$, where $A_0(z)$ (respectively $A_1(z)$) is compact and continuous in $z < 0$ (respectively $z \leqslant 0$). Assume that for some function $f(\cdot)$, $f(z) \to 0$, $z \to 0-$ one has*

$$\lim_{z \to 0-} f(z) n(\lambda, A_0(z)) = l(\lambda),$$

and $l(\lambda)$ *is continuous in* $\lambda > 0$. *Then the same limit exists for* $A(z)$ *and*

$$\lim_{z\to 0-} f(z)n(\lambda, A(z)) = l(\lambda).$$

Remark 9.3. According to lemma 9.2 any perturbations of the operator $A_0(z)$ defined in lemma 9.2, which is compact and continuous up to $z = 0$, do not contribute to the asymptotics (9.1). Throughout the proof of theorem 9.1, we shall use this fact without further comments.

Lemma 9.4. *There exists* $\delta > 0$ *such that*

$$E(0, \gamma; p, q) = \tfrac{1}{2}[(1+\gamma)|p|^2 + 2\gamma(p, q) + (1+\gamma)|q|^2] + O(|p|^4 + |q|^4) \tag{9.2}$$

as $p, q \to 0$ *and for all* $z \in (-\delta, 0]$

$$\Delta_\gamma(0, p, z) = \frac{\mu(\gamma)(\gamma)^{\frac{3}{2}}}{2\pi(1+\gamma)^{\frac{3}{2}}}(n_\gamma|p|^2 - 2z)^{\frac{1}{2}} + O(|p|^2 + |z|) \qquad \text{as} \quad p, z \to 0, \tag{9.3}$$

where $n_\gamma = (1+2\gamma)(1+\gamma)^{-1}$.

Proof. The asymptotics

$$\varepsilon(p) = \tfrac{1}{2}|p|^2 + O(|p|^4) \qquad \text{as} \quad p \to 0 \tag{9.4}$$

of the function $\varepsilon(p)$ yields (9.2). The definition of $\mathcal{E}_{\min,\gamma}(k)$ and the representation (5.3) give the asymptotics

$$\mathcal{E}_{\min,\gamma}(k) = \frac{\gamma}{2(1+\gamma)}|k|^2 + O(|k|^4) \qquad \text{as} \quad k \to 0, \tag{9.5}$$

which yields (9.3). □

Denote by $\chi_\delta(\cdot)$ the characteristic function of $U_\delta(0) = \{p \in \mathbb{T}^3 : |p| < \delta\}$. Let $T\left(\delta, \frac{|K|^2}{2M_\gamma} + |z|\right)$ be an operator on $L^2(\mathbb{T}^3)$ with the kernel

$$-D_\gamma \frac{\chi_\delta(p)\chi_\delta(q)\left(n_\gamma|p|^2 + 2\left(\frac{|K|^2}{2M_\gamma} + |z|\right)\right)^{-1/4}\left(n_\gamma|q|^2 + 2\left(\frac{|K|^2}{2M_\gamma} + |z|\right)\right)^{-1/4}}{(1+\gamma)|q|^2 + 2\gamma(p, q) + (1+\gamma)|p|^2 + 2\left(\frac{|K|^2}{2M_\gamma} + |z|\right)},$$

where

$$D_\gamma = \frac{(1+\gamma)^{\frac{3}{2}}}{2\pi^2}, \qquad n_\gamma = \frac{1+2\gamma}{1+\gamma}, \qquad M_\gamma = \frac{1+2\gamma}{\gamma}.$$

Lemma 9.5. *The operator* $T_\gamma(K, z) - T_\gamma\left(\delta; \frac{|K|^2}{2M_\gamma} + |z|\right)$ *belongs to the Hilbert–Schmidt class and is continuous in* $K \in \mathbb{T}^3$ *and* $z \leqslant 0$.

Proof. Applying the asymptotics (9.2) and (9.3), one can estimate the kernel of the operator $T_\gamma(K, z) - T_\gamma(\delta; \frac{|K|^2}{2M_\gamma} + |z|)$ by

$$C[(|p|^2 + |q|^2)^{-1} + |p|^{-\frac{1}{2}}(|p|^2 + |q|^2)^{-1} + (|q|^{-\frac{1}{2}}(|p|^2 + |q|^2)^{-1} + 1],$$

and hence the operator $T_\gamma(K, z) - T_\gamma\left(\delta; \frac{|K|^2}{2M_\gamma} + |z|\right)$ belongs to the Hilbert–Schmidt class for all $K \in U_\delta(0)$ and $z \leqslant 0$. In combination with the continuity of the kernel of the operator in $K \in U_\delta(0)$ and $z < 0$, this gives the continuity of $T_\gamma(K, z) - T_\gamma\left(\delta; \frac{|K|^2}{2M_\gamma} + |z|\right)$ in $K \in U_\delta(0)$ and $z \leqslant 0$. □

Let

$$\mathbf{S}_\gamma(\mathbf{r}) : L^2((0,\mathbf{r}),\sigma_0) \to L^2((0,\mathbf{r}),\sigma_0), \qquad \mathbf{r} = 1/2|\log\left(\frac{|K|^2}{2M_\gamma} + |z|\right), \quad \sigma_0 = L^2(\mathbb{S}^2),$$

$\mathbb{S}^2$ being the unit sphere in $\mathbb{R}^3$, be the integral operator with the kernel

$$S_\gamma(t;\, y) = (2\pi)^{-2}\frac{u_\gamma}{\cos hy + s_\gamma t}, \tag{9.6}$$

$$\begin{aligned} u_\gamma &= \frac{1+\gamma}{\sqrt{1+2\gamma}}, & s_\gamma &= \frac{\gamma}{1+\gamma}, \\ y &= x - x\prime, x, x\prime \in (0,\mathbf{r}), & t &= \langle \xi, \eta\rangle, \quad \xi,\eta \in \mathbb{S}^2, \end{aligned} \tag{9.7}$$

and let

$$\hat{\mathbf{S}}_\gamma(\lambda) : \sigma_0 \to \sigma_0, \lambda \in (-\infty, +\infty)$$

be the integral operator with the following kernel:

$$\hat{S}_\gamma(t;\lambda) = \int_{-\infty}^{+\infty} \exp\{-\mathrm{i}\lambda r\} S_\gamma(t; r)\,\mathrm{d}r = -(2\pi)^{-1}u_\gamma \frac{\sinh[\lambda(arc\cos s_\gamma t)]}{\left(1 - s_\gamma^2 t^2\right)^{\frac{1}{2}} \sinh(\pi\lambda)}. \tag{9.8}$$

For $\mu > 0$, define

$$U(\mu,\gamma) = (4\pi)^{-1}\int_{-\infty}^{+\infty} n(\mu, \hat{\mathbf{S}}_\gamma(y))\,\mathrm{d}y. \tag{9.9}$$

Lemma 9.6. *The function $U(\mu;\gamma)$ is continuous in $\mu > 0$; the following limit*

$$\lim_{\mathbf{r}\to\infty} \tfrac{1}{2}\mathbf{r}^{-1} n(\mu, \mathbf{S}_\gamma(\mathbf{r})) = U(\mu;\gamma)$$

exists.

Remark 9.7. This lemma can be proven quite similarly to the corresponding results of [8]. In particular, the continuity of $U(\mu;\gamma)$ in $\mu > 0$ is a result of lemma 3.4; theorem 4.5 states the existence of the limit

$$\lim_{\mathbf{r}\to\infty} \tfrac{1}{2}\mathbf{r}^{-1} n(\mu, \mathbf{S}_\gamma(\mathbf{r}) = U(\mu,\gamma).$$

Lemma 9.8. *The equality*

$$\lim_{\frac{|K|^2}{M_\gamma}+|z|\to 0} \frac{n\left(1, T_\gamma\left(\delta, \frac{|K|^2}{2M_\gamma} + |z|\right)\right)}{\left|\log\left(\frac{|K|^2}{M_\gamma} + |z|\right)\right|} = U(\gamma) \tag{9.10}$$

holds.

Proof. The space of functions having support in $U_\delta(0)$ is an invariant subspace for the operator $T_\gamma\left(\delta, \frac{|K|^2}{2M_\gamma} + |z|\right)$.

Let $T_\gamma^{(0)}\left(\delta, \frac{|K|^2}{2M_\gamma} + |z|\right)$ be the restriction of the operator $T_\gamma\left(\delta, \frac{|K|^2}{2M_\gamma} + |z|\right)$ on the invariant subspace $L^2(U_\delta(0))$.

The operator $T_\gamma^{(0)}\left(\delta, \frac{|K|^2}{2M_\gamma} + |z|\right)$ is unitarily equivalent with the operator $T_\gamma^{(1)}\left(\delta, \frac{|K|^2}{2M_\gamma} + |z|\right)$ acting in $L^2(U_r(0))$ by

$$T_\gamma^{(1)}\left(\delta;\, \frac{|K|^2}{2M_\gamma} + |z|\right) w(p) = -D_\gamma \int_{U_r(0)} \frac{(n_\gamma|p|^2+2)^{-1/4}(n_\gamma|q|^2+2)^{-1/4}}{(1+\gamma)|p|^2 + 2\gamma(p,q) + (1+\gamma)|q|^2 + 2} w(q)\,\mathrm{d}q,$$

where $B_r = \left\{p \in T^3 : |p| < r, r = \left(\frac{|K|^2}{2M_\gamma} + |z|\right)^{-\frac{1}{2}}\right\}$.

The equivalence is performed by the unitary dilation

$$U_r : L^2(U_\delta(0)) \to L^2(B_r), \qquad (U_r f)(p) = \left(\frac{r}{\delta}\right)^{-3/2} f\left(\frac{\delta}{r}p\right).$$

Denote by $\chi_1(\cdot)$ the characteristic function of $U_1(0)$. We may replace

$$(n_\gamma|p|^2+2)^{-1/4}, \qquad (n_\gamma|q|^2+2)^{-1/4}$$

and

$$(1+\gamma)|p|^2+2\gamma(p,q)+(1+\gamma)|q|^2+2$$

by

$$(n_\gamma|p|^2)^{-1/4}(1-\chi_1(p)), \qquad (n_\gamma|q|^2)^{-1/4}(1-\chi_1(q))$$

and

$$(1+\gamma)|p|^2+2\gamma(p,q)+(1+\gamma)|q|^2,$$

respectively, since the error will be a Hilbert–Schmidt operator continuous up to $\frac{|K|^2}{2M_\gamma}+|z|=0$.

Then we get the operator $T_\gamma^{(2)}(r)$ in $L^2(U_r(0)\backslash U_1(0))$ with the kernel

$$-D_\gamma\sqrt{\frac{1+\gamma}{\sqrt{1+2\gamma}}}\frac{|p|^{-1/2}|q|^{-1/2}}{(1+\gamma)|p|^2+2\gamma(p,q)+(1+\gamma)|q|^2}.$$

By the dilation

$$\mathbf{M} : L^2(U_r(0)\backslash U_1(0)) \longrightarrow L^2((0,\mathbf{r})\times\sigma_0), \qquad r = 1/2\left|\log\left(\frac{|K|^2}{2M_\gamma}+|z|\right)\right|,$$

where $(Mf)(x,w) = \mathrm{e}^{3x/2}f(\mathrm{e}^x w)$, $x\in(0,\mathbf{r})$, $w\in\mathbb{S}^2$, one sees that the operator $T_\gamma^{(2)}(r)$ is unitary equivalent to the integral operator $\mathbf{S}_\gamma(\mathbf{r})$. The difference of the operators $\mathbf{S}_\gamma(\mathbf{r})$ and $T_\gamma\big(\delta, \frac{|K|^2}{2M_\gamma}+|z|\big)$ is compact (up to unitarily equivalence), and hence lemma 9.6 yields the equality 9.10. Lemma 9.8 is thus proved. □

Lemma 9.9. *For any $\gamma>0$, the inequality $U(\gamma)>0$ holds.*

Proof. It is convenient to calculate the coefficient $U(\gamma)$ by means of a decomposition of the operator $\hat{\mathbf{S}}_\gamma(y)$ into the orthogonal sum over its invariant subspaces.

Denote by $L_l\subset L^2(S^2)$ the subspace of the harmonics of degree $l=0,1,\ldots$. It is clear that $L^2(S^2)=\sum_{l=0}^{\infty}\oplus L_l$, $\dim L_l = 2l+1$. Let $P_l : L^2(S^2)\to L_l$ be the orthogonal projector onto L_l.

The kernel of P_l is expressed via the Legendre polynomial $P_l(\cdot)$:

$$P_l(\xi,\eta) = \frac{2l+1}{4\pi}P_l(\langle\xi,\eta\rangle).$$

The kernel of $\hat{\mathbf{S}}_\gamma(y)$ depends on the scalar product $\langle\xi,\eta\rangle$ only, so that the subspaces L_l are invariant for $\hat{\mathbf{S}}_\gamma(y)$ and

$$\hat{\mathbf{S}}_\gamma(y) = \sum_{l=0}^{\infty}\oplus\big(\hat{\mathbf{S}}_\gamma^{(l)}(y)\otimes P_l\big), \tag{9.11}$$

where $\hat{\mathbf{S}}_\gamma^{(l)}(y)$ is the multiplication operator by the number

$$\hat{\mathbf{S}}_\gamma^{(l)}(y) = 2\pi\int_{-1}^{1}P_l(t)\hat{\mathbf{S}}_\gamma(t;y)\,\mathrm{d}t \tag{9.12}$$

in L_l, the subspace of the harmonics of degree l, and $P_l(t)$ is a Legendre polynomial. Therefore,

$$n(\mu, \hat{\mathbf{S}}_\gamma(y)) = \sum_{l=0}^{\infty}(2l+1)n\big(\mu, \hat{\mathbf{S}}_\gamma^{(l)}(y)\big), \qquad \mu > 0$$

It follows from (9.9) and (9.12) that

$$U(1,\gamma) \geqslant \frac{1}{4\pi}\int_{-\infty}^{+\infty} n\big(1, \hat{\mathbf{S}}_\gamma^{(0)}(y)\big)\,\mathrm{d}y \geqslant \frac{1}{4\pi}\mathrm{mes}\big\{x : \hat{\mathbf{S}}_\gamma^{(0)}(x) > 1\big\}. \tag{9.13}$$

By (9.8), we first calculate $\hat{\mathbf{S}}_\gamma^{(0)}(y)$:

$$\hat{\mathbf{S}}_\gamma^{(0)}(y) = \frac{u_\gamma}{\sinh(\pi\gamma)}\int_{-1}^{1}\frac{\sinh[y(\arccos(s_\gamma(t))]}{\sqrt{1-s_\gamma^2t^2}}\,\mathrm{d}t. \tag{9.14}$$

Applying the equality

$$\frac{u_\gamma \sinh[y(\arcsin s_\gamma)]}{s_\gamma y\cosh\left(\frac{\pi y}{2}\right)} = \frac{u_\gamma}{\sinh(\pi y)}\int_{-1}^{1}\frac{\sinh[y(\arccos(s_\gamma t))]}{\sqrt{1-s_\gamma^2t^2}}\,\mathrm{d}t, \qquad y\in\mathbb{R},$$

and using

$$\frac{\sinh[y(\arcsin s_\gamma)]}{s_\gamma y} \geqslant, 1$$

we have

$$\max_y \hat{S}_\gamma^{(0)}(y) \geqslant \frac{u_\gamma}{\cosh\left(\frac{\pi y}{2}\right)} \geqslant u_\gamma > 1.$$

This together with (9.13) completes the proof. □

Proof of theorem 9.1. Lemmas 9.2, 8.4, 9.5, 9.8 and 9.9 yield the proof of theorem 9.1. □

Proof of theorem 4.9. The proof of theorem 4.9 concerning the infiniteness of eigenvalues and the asymptotics follows from theorems 7.2 and 9.1 □

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